

Binder Parameter of a Heisenberg Spin-Glass Model in Four Dimensions

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We studied the phase transition of the $\pm J$ Heisenberg model with and without a random anisotropy on four dimensional lattice $L \times L \times L \times (L+1)$ ($L \leq 9$). We showed that the Binder parameters $g(L, T)$'s for different sizes do not cross even when the anisotropy is present. On the contrary, when a strong anisotropy exists, $g(L, T)$ exhibits a steep negative dip near the spin-glass phase transition temperature T_{SG} similarly to the p -state infinite-range Potts glass model with $p \geq 3$, in which the one-step replica-symmetry-breaking (RSB) occurs. We speculated that a one-step RSB-like state occurs below T_{SG} , which breaks the usual crossing behavior of $g(L, T)$.

Recently, the low temperature phase of the XY and Heisenberg spin-glass (SG) models has been attacked a great interest. It is known that vector SG models have a chiral symmetry in addition to the continuous symmetry¹. Consequently, these models may have both SG order and chiral glass (CG) order. Kawamura and coworkers claimed that decoupling of the spin and chiral valuables occurs at long distances^{2,3}, and that the CG order is realized at low temperatures in three dimensions ($d = 3$), but the SG order is not^{4,5,6,7}. Although the existence of the CG order has been accepted, controversy exists on the SG order. Maucourt and Grempel⁸ studied the domain wall energy $W(L)$ of the $\pm J$ XY model on finite $d = 3$ lattices of L^3 at absolute zero temperature ($T = 0$) and speculated that $W(L)$ may increase with the linear size L . The same speculation was also given by Kosterlitz and Akino⁹. The present authors examined the stiffness of the $\pm J$ Heisenberg model at $T = 0$ ¹⁰ and $T \neq 0$ ¹¹, and suggested that the stiffness exponent θ in $d = 3$ has a positive value for $T/J \lesssim 0.19$. They also performed a Monte-Carlo (MC) simulation of the same model and found that the SG susceptibility χ_{SG} exhibits a divergent behavior toward the temperature of $T/J \sim 0.18$ ¹². These facts strongly suggests that the SG order also occurs in the XY and Heisenberg models in $d = 3$. This view of the SG order was supported in recent studies of the aging effect of the spin autocorrelation function $\langle \mathbf{S}_i(0) \mathbf{S}_i(t) \rangle$ ¹³ and the non-equilibrium relaxation of χ_{SG} ¹⁴. However, the Binder parameters $g(L, T)$'s for different L neither cross nor come together in both the XY⁷ and Heisenberg⁶ SG models.

This fact raises an objection against the occurrence of the SG order, because it is believed that the most reliable evidence of the occurrence of the SG phase transition is the crossing of $g(L, T)$'s at the same temperature of T_{SG} ^{15,16,17}. In fact, it was reported that the crossing of $g(L, T)$'s occurs in both the XY¹⁸ and Heisenberg¹⁹ SG models in four dimensions ($d = 4$). Thus we are faced by a serious problem that *we have gotten the opposite views of the SG order from the studies of the different*

quantities.

The problem may arise from a poor knowledge of the property of $g(L, T)$ of the vector SG model. In the non-frustrated system, the crossing of $g(L, T)$'s occurs at T_{C} with some positive value of \tilde{g} ($\equiv g(L, T_{\text{C}})$). This property results from the fact that, in the thermodynamic limit, $g(\infty, T) = 0$ for $T > T_{\text{C}}$ and $g(\infty, T) = 1$ for $T < T_{\text{C}}$. However, the same will not always be true in the SG model, because $g(\infty, T)$ for $T < T_{\text{SG}}$ will take different values due to the occurrence of the replica symmetry breaking (RSB). In fact, it was revealed by Hukushima and Kawamura²⁰ that $g(\infty, T_{\text{SG}}^-)$ of the infinite-range p -state Potts glass model²¹ takes different values depending on the state number p , where $T^- = \lim_{\epsilon \rightarrow 0} (T - \epsilon)$. Therefore it is crucially important to reveal the property of $g(L, T)$ in such vector SG models in which the SG phase transition occurs at $T_{\text{SG}} \neq 0$.

In this paper, we report that $g(L, T)$ of the Heisenberg SG model in $d = 4$ exhibits a behavior quite different from that of the Ising SG model. We reexamined $g(L, T)$ of the $\pm J$ Heisenberg models in $d = 4$ with and without a random anisotropy of magnitude D . In the case of $D = 0$, when L is increased, the increment of $g(L, T)$ with decreasing temperature becomes less steep at low temperatures, and the crossing of $g(L, T)$'s which was suggested for small L ¹⁹ is released. When $D \neq 0$, $g(L, T)$ for each L decreases, particularly around T_{SG} , and this property is enhanced when D is increased, where T_{SG} is the SG transition temperature estimated from the scaling plot of χ_{SG} . In particular, $g(L, T)$ for large D exhibits a negative dip near T_{SG} which deepens with increasing L . These facts are quite interesting, because the anisotropy would stabilize the SG order^{22,23}. We believe, hence, that *the absence of the crossing of $g(L, T)$'s for finite L says nothing about the presence of the SG order in this model.* We speculate that, even for $D = 0$, $g(\infty, T_{\text{SG}}^-)$ takes a small positive value (or a negative value) due to the occurrence of a one-step RSB-like state. We hope our findings will help to understand the low temperature phase of the Heisenberg SG model in $d = 3$.

We start with the anisotropic $\pm J$ Heisenberg model on a hyper cubic lattice of $L \times L \times L \times (L+1) (\equiv N)$ with skew boundary conditions along three L directions and a periodic boundary condition along the $(L+1)$ direction. The Hamiltonian is described by

$$H = - \sum_{\langle ij \rangle} [J_{ij} \mathbf{S}_i \mathbf{S}_j + \sum_{\alpha \neq \beta} D_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta], \quad (1)$$

where \mathbf{S}_i is the Heisenberg spin of $|\mathbf{S}_i| = 1$ and S_i^α is its α -component ($\alpha = x, y, z$), and $\langle ij \rangle$ runs over all nearest-neighbor pairs. The exchange interaction J_{ij} takes on either $+J$ or $-J$ with the same probability of $1/2$. We assume that the anisotropy comes from pseudo-dipolar couplings and impose the restriction $D_{ij}^{\alpha\beta} = D_{ji}^{\alpha\beta} = D_{ij}^{\beta\alpha}$. We further assume that $D_{ij}^{\alpha\beta}$ are uniform random values between $-D$ and D . We note that the role of the anisotropy is to break the rotational symmetry of the model and to stabilize the SG order.

We performed a MC simulation of the two-replica systems of $\{\mathbf{S}_i\}$ and $\{\tilde{\mathbf{S}}_i\}$ using an exchange MC algorithm²⁴. We calculated the order-parameter probability distribution $P_L(q)$ of

$$P_L(q) = [\langle \delta(q - Q) \rangle], \quad (2)$$

where $\langle \dots \rangle$ and $[\dots]$ mean the thermal average and the bond distribution average, respectively. Here Q is the spin overlap defined by

$$Q = \sqrt{\frac{1}{3} \sum_{\alpha, \beta} (q^{\alpha\beta})^2}, \quad (3)$$

with $q^{\alpha\beta} \equiv \frac{1}{N} \sum_{i=1}^N S_i^\alpha \tilde{S}_i^\beta$. Using $P_L(q)$, we obtained the SG susceptibility χ_{SG} and the Binder parameter $g(L, T)$ which are defined by

$$\chi_{SG} = 3N[\langle q^2 \rangle], \quad (4)$$

$$g(L, T) = \frac{1}{2} (11 - 9 \frac{[\langle q^4 \rangle]}{[\langle q^2 \rangle]^2}), \quad (5)$$

where $[\langle q^n \rangle] = \int q^n P_L(q) dq$. To examine whether the SG order occurs or not, we also calculated quantities $A(L, T)$ and $G(L, T)$ that measure the order-parameter fluctuations (OPF):

$$A(L, T) = \frac{[\langle q^2 \rangle^2] - [\langle q^2 \rangle]^2}{[\langle q^2 \rangle]^2}, \quad (6)$$

$$G(L, T) = \frac{[\langle q^2 \rangle^2] - [\langle q^2 \rangle]^2}{[\langle q^4 \rangle] - [\langle q^2 \rangle]^2}. \quad (7)$$

Each of these quantities will exhibits a crossing behavior at T_{SG} when the SG transition occurs^{25,26}. We performed the MC simulation of the model (1) with various values of $D = 0, 0.1J, 0.2J, 0.5J$ and $1.0J$. The linear sizes of the lattice studied here are $L = 3 \sim 9$. Equilibration was checked by monitoring the stability of the results against

at least two-times longer runs. The numbers of the samples were 480 for $L = 3$, 288 for $L = 5$, 96 for $L = 7$, and 48 for $L = 9$.

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FIG. 1. (a) Temperature dependences of the spin-glass susceptibility χ_{SG} for different sizes of the lattice. Open symbols represent those for $D = 0$ and closed symbols for $D = 0.5J$. (b) Plot of χ_{SG} for $D = 0$.

In Fig. 1(a), we show results of χ_{SG} in $D = 0$ and $D = 0.5J$. In both the cases, χ_{SG} for larger L increases rapidly as the temperature is decreased. The same was true for different values of D . The finite size scaling analysis in each D suggested the divergence of χ_{SG} for $L \rightarrow \infty$ at a finite, non-zero temperature. An example of the scaling plot in the case of $D = 0$ is presented in Fig. 1(b). Hereafter, we tentatively call this temperature the SG transition temperature and denote T_{SG} . Note that T_{SG} increases with increasing D . These results are compatible with a common belief that the SG order occurs in $d = 4$ even for $D = 0$, and that the SG order is stabilized by the anisotropy^{22,23}.

Now we show $g(L, T)$ in different D 's in Figs. 2(a) - 2(c). When $D = 0$, $g(L, T)$'s for $L = 3$ and 5 come close to each other near T_{SG} . But they do not cross, because the increment of $g(L, T)$ for $L = 5$ is suppressed below T_{SG} . This suppression is not released for larger L . Therefore we believe that, in contrary to the previous suggestion¹⁹, $g(L, T)$'s for large L do not cross but converge on some non-zero value \tilde{g} at $T = T_{SG}$.²⁷ When $D \neq 0$, the suppression is enhanced more. In particular, for large D , $g(L, T)$ exhibits a dip near T_{SG} which deepens as L is increased. This result is also incompatible with our naive expectation that, as D is increased, $g(L, T)$'s would tend to cross with some positive \tilde{g} , because the anisotropy will stabilize the SG order.

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FIG. 2. Temperature dependences of the Binder parameter $g(L, T)$ of different sizes of the lattice; (a) $D = 0$ and $D = 0.1J$ (open symbols), (b) $D = 0.2J$ and $D = 0.5J$ (open symbols), (c) $D = 0.5J$ and $D = 1.0J$ (closed symbols). Arrows indicate the SG transition temperature T_{SG} .

We next examined $G(L, T)$ which would exhibit a crossing behavior at T_{SG} even when $g(L, T)$ did not exhibit the crossing behavior^{20,26}. Here, in Figs. 3(a) and 3(b), we show $G(L, T)$'s for different L in the cases of $D = 0$ and $D = 0.5J$, respectively. When $D = 0$ (and also $D \leq 0.2J$), $G(L, T)$'s for large $L (\geq 5)$ seem to come together near T_{SG} . This property becomes more prominent in large D , and the data for $L = 3$ joins. A similar crossing has also been found in the other quantity $A(L, T)$.

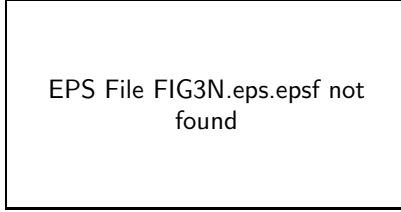


FIG. 3. Temperature dependences of the order-parameter fluctuation (OPF) $G(L, T)$ of the $\pm J$ Heisenberg model in $d = 4$ for different sizes of the lattice and for different magnitude of the anisotropy: (a) $D = 0$ and (b) $D = 0.5J$. Arrows indicate the SG transition temperature T_{SG} estimated from the scaling plots of the Binder parameter $g(L, T)$, but from the crossing of $G(L, T)$'s (and also $A(L, T)$'s), as well as from the divergence of the SG susceptibility χ_{SG} . This fact was quite interesting, because it threw a doubt on the common belief that the most reliable evidence of the occurrence of the SG order is the crossing of $g(L, T)$'s. We speculated that the low temperature phase of the vector SG model is characterized by one-step RSB-like state and the crossing of $g(L, T)$'s for finite L is absent.

We have calculated four quantities χ_{SG} , $g(L, T)$, $G(L, T)$ and $A(L, T)$ to examine the SG phase transition. All the quantities except for $g(L, T)$ suggested the occurrence of the SG order below T_{SG} . Considering the facts that $G(L, T)$ and $A(L, T)$ can be used to determine the value of T_{SG} of such SG models in which T_{SG} is hardly determined by the usual crossing behavior of $g(L, T)$ ^{20,26}, and that the L -dependence of χ_{SG} clearly suggests the divergence of the spin correlation length at T_{SG} , it is natural to conclude that the SG order occurs below T_{SG} . Therefore, the absence of the crossing of $g(L, T)$'s for finite L would say nothing about the SG phase transition in this model.

What occurs in $g(L, T)$ at T_{SG} for $L \rightarrow \infty$? The occurrence of a dip of $g(L, T)$ near T_{SG} for large D ($D = 0.5J$ and $1.0J$) is suggestive. The dip deepens as L is increased implying a negative divergence of $g(\infty, T_{\text{SG}}^-)$. It is known that, in the infinite-range p -state Potts glass, the value of $g(\infty, T_{\text{PG}}^-)$ changes continuously from 1 for $p = 2$ (Ising model) to $-\infty$ for $p = 4$ due to the occurrence of the one-step RSB for $p > 2$ ^{20,21}. Therefore we suggest that $g(\infty, T)$ takes $-\infty$ or some very large negative value at T_{SG}^- due to the occurrence of a one-step RSB-like state. In fact, as shown in Fig. 4, the order parameter distribution function $P_L(q)$ for $D = 1.0J$ exhibits a double peak at low temperatures²⁸.

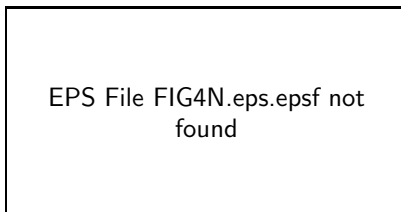


FIG. 4. The order-parameter probability distribution $P_L(q)$ of the $\pm J$ Heisenberg model at $T = 0.6J (< T_{\text{SG}} \sim 0.95J)$. The $\pm J$ Heisenberg model at $T = 0.6J$ is not a mat/0009292.

On the other hand, in the case of small D , no distinct dip but a bending of $g(L, T)$ was seen around T_{SG} . We may also explain this behavior based on a plausible assumption that $g(\infty, T_{\text{SG}}^-)$ takes a negative value or a small positive value due to the occurrence of the one-step RSB-like state. When $D = 0$, the rotational symmetry recovers and $g(\infty, T)$ might exhibit some different property. We think, however, that $g(\infty, T_{\text{SG}}^-)$ also takes a small positive value (or a negative value), because our data of Figs. 2(a) - 2(c) imply the presence of no gap between the cases of $D = 0$ and $D \neq 0$. Of course, we could not rule out the possibility that there exists some threshold D^* , including $D^* = 0$, below which the nature of $g(\infty, T)$ changes qualitatively.

In summary, we reexamined the spin-glass phase transition of the $\pm J$ Heisenberg model in four dimensions ($d = 4$) and gave a confirmation that the SG transition really occurs even when the anisotropy is absent $D = 0$. However, its transition temperature T_{SG} could not be determined from the usual crossing behavior of the Binder parameter $g(L, T)$, but from the crossing of $G(L, T)$'s (and also $A(L, T)$'s), as well as from the divergence of the SG susceptibility χ_{SG} . This fact was quite interesting, because it threw a doubt on the common belief that the most reliable evidence of the occurrence of the SG order is the crossing of $g(L, T)$'s. We speculated that the low temperature phase of the vector SG model is characterized by one-step RSB-like state and the crossing of $g(L, T)$'s for finite L is absent.

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- ²⁷ The crossing behavior of $g(L, T)$ at $T = T_{\text{SG}}$ comes from the combination of (A) the scaling hypothesis and (B) the assumption $g(\infty, T) = 0$ for $T > T_{\text{SG}}$ and $g(\infty, T) = 1$ for $T < T_{\text{SG}}$. Therefore when the SG phase transition occurs, even if the assumption (B) is not satisfied, $g(L, T)$ will converge on some non-zero value at $T = T_{\text{SG}}$. We think a large finite size effect masks this property.
- ²⁸ Note that a double peak was found in the chirality order parameter distribution $P_L^{(c)}(q)$ of the Heisenberg⁶ and XY⁷ SG models with $D = 0$ in $d = 3$ in which the one-step RSB was suggested to occur.

